

# A new class of codes for Boolean masking of cryptographic computations

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## Abstract

We introduce a new class of rate one-half binary codes: **complementary information set codes**. A binary linear code of length  $2n$  and dimension  $n$  is called a complementary information set code (CIS code for short) if it has two disjoint information sets. This class of codes contains self-dual codes as a subclass. It is connected to graph correlation immune Boolean functions of use in the security of hardware implementations of cryptographic primitives. Such codes permit to improve the cost of masking cryptographic algorithms against side channel attacks. In this paper we investigate this new class of codes: we give optimal or best known CIS codes of length  $< 132$ . We derive general constructions based on cyclic codes and on double circulant codes. We derive a Varshamov-Gilbert bound for long CIS codes, and show that they can all be classified in small lengths  $\leq 12$  by the building up construction. Some nonlinear permutations are constructed by using  $\mathbb{Z}_4$ -codes, based on the notion of dual distance of an unrestricted code.

**Keywords:** cyclic codes, self-dual codes, dual distance, double circulant codes,  $\mathbb{Z}_4$ -codes

## I. INTRODUCTION

Since the seminal work of P. Kocher [18], [19], it is known that the physical implementation of cryptosystems on devices such as smart cards leaks information which can be used in differential power analysis or in other kinds of side channel attacks. These attacks can be devastating if proper counter-measures are not included in the implementation. A kind of counter-measure, which is suitable for both hardware and software cryptographic implementations and does not rely on specific hardware properties is the following. It consists in applying a kind of secret sharing method, changing the variable representation (say  $x$ ) into randomized shares  $m_1, m_2, \dots, m_{d+1}$  called masks such that  $x = m_1 + m_2 + \dots + m_{d+1}$  where  $+$  is a group operation - in practice, the XOR. Since the difficulty of performing an attack of order  $d$  (involving  $d + 1$  shares) increases exponentially with  $d$ , it was believed until recently that for increasing the resistance to attacks, new masks have to be added, thereby increasing the order of the countermeasure, see [25]. But in these schemes, the profusion of masks implies a heavy load on the true random number generator, which significantly degrades the performance of the device. Moreover, the solution in [25] bases itself on a mask refreshing operation for which no secure implementation has been detailed so far. Now, it is shown in [21] that another option consists in encoding some of the masks, which is much less costly than adding fresh masks. At the order 1, this consists in representing  $x$  by the ordered pair  $(F(m), x + m)$ , where  $F$  is a well

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chosen permutation, rather than  $(m, x+m)$ . The encoding can even be free, in the case it can be merged in a single table with the downstream function. Notably, it is demonstrated in [22], [23] that the same effect as adding several masks can be obtained by the encoding of one single mask.

This method, called **leakage squeezing**, uses permutations  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ , such that, given some integer  $d$  as large as possible, for every pair of vectors  $a, b \in \mathbb{F}_2^n$  such that the concatenated vector  $(a, b)$  is nonzero and has Hamming weight  $< d$ , the value of the Walsh-Hadamard transform of  $F$  at  $(a, b)$ , is null. We call such functions  $d$ -**GCI**, for **Graph Correlation Immune** of order  $d$  since the condition is equivalent to saying that the indicator function of the graph  $\{(x, F(x)); x \in \mathbb{F}_2^n\}$  is correlation immune of order  $d-1$  [5]. Thus a  $d$ -GCI function is a protection against an attack of order  $d-1$ . These functions were introduced in [22] where it is proved that the existence of such Boolean functions when they are **linear** is equivalent to the existence of binary linear codes with parameters  $[2n, n, \geq d]$  having two disjoint information sets. Based on this equivalence we say that a binary linear code of length  $2n$  and dimension  $n$  is **Complementary Information Set** (CIS for short) with a partition  $L, R$  if there is an information set  $L$  whose complement  $R$  is also an information set. In general, we will call the partition written in pidgin Maple as  $[1..n], [n+1..2n]$  the **systematic partition**. More explicitly, we describe CIS codes with relation to permutations as follows.

Assuming a systematic possibly nonlinear code  $C$  of length  $2n$  of the form

$$C = \{(x, F(x)) \mid x \in \mathbb{F}_2^n\},$$

the permutation is constructed as the map  $x \mapsto F(x)$ . In that setting  $C$  is CIS by definition if and only if  $F$  is a bijection. When  $C$  is a linear code, we can also consider a systematic generator matrix  $(I, A)$  of the code, where  $I$  is the identity matrix of order  $n$  and  $A$  is a square matrix of order  $n$ . Then  $F(x) = xA$ , and the CIS condition reduces to the fact that  $A$  is nonsingular.

On the other hand, since the complement of an information set of a  $[2n, n]$  linear code is an information set for its dual code, it is clear that systematic **self-dual codes** are CIS with the systematic partition. It is also clear that the dual of a CIS code is CIS. Hence CIS codes are a natural generalization of self-dual codes.

In the present work we give explicit constructions of optimal CIS codes of length  $< 132$ , and derive a Varshamov-Gilbert bound for long CIS codes<sup>1</sup>. We give general constructions based on cyclic codes and double circulant codes. We show that all CIS codes of length  $\leq 12$  can be classified by the building up construction, an important classification tool for self-dual codes [16]. We go back to the notion of graph correlation immune function, and based on the notion of dual distance we give a class of unrestricted codes giving Boolean functions with immunity of higher degree than the functions in the same number of variables obtained from linear CIS codes (the practical performance of non linear bijections is explored in the companion paper [23]). In particular the best codes we obtain in that way are binary images of certain  $\mathbb{Z}_4$ -codes, the quadratic residue codes [2]. Notice moreover that the notion of being CIS is not trivial since there exists at least one optimal code (a  $[34, 17, 8]$  code) which is not CIS.

The material is organized as follows. Section 2 collects the necessary notation and definitions. Section 3 extends the notion of CIS codes to possibly nonlinear codes. Section 4 contains the general constructions of CIS codes using cyclic codes and double circulant codes. Section 5

<sup>1</sup>While in cryptographic practice  $n$  is not so large, the coding problem is of sufficient intrinsic interest to motivate the study of long codes.

describes the known optimal linear CIS codes of length  $< 132$ . Section 6 is dedicated to the building up construction and gives a classification of CIS codes of length  $\leq 12$ . Section 7 derives a VG bound on long linear CIS codes without relying on previous knowledge on the asymptotic performance of self-dual codes. All our computations were done using Magma [26].

## II. NOTATION AND DEFINITIONS

### A. Binary codes

Let  $C$  be a binary linear code. Its parameters are formatted as  $[n, k, d]$  denoting length, dimension, and minimum distance. By an unrestricted code we shall mean a possibly nonlinear code. The dual  $C^\perp$  of a linear code  $C$  is understood to be with respect to the standard inner product. The code  $C$  is **self-dual** if  $C = C^\perp$  and **isodual** if  $C$  is equivalent to  $C^\perp$ . A self-dual code is **Type II** if the weight of each codeword is a multiple of four and **Type I** otherwise. A binary (unrestricted) code  $C$  of length  $n$  is called **systematic** if there exists a subset  $I$  of  $\{1, \dots, n\}$  called an **information set** of  $C$ , such that every possible tuple of length  $|I|$  occurs in exactly one codeword within the specified coordinates  $x_i$ ;  $i \in I$ . We call **CIS (unrestricted) code** a systematic code which admits two complementary information sets. The **Hamming weight**  $w(z)$  of a binary vector  $z$  is the number of its nonzero entries. The **weight enumerator**  $W_C(x, y)$  of a code  $C$  is the homogeneous polynomial defined by

$$W_C(x, y) = \sum_{c \in C} x^{n-w(c)} y^{w(c)}.$$

The code  $C$  is **formally self-dual** or fsd for short, if its weight enumerator is invariant under the MacWilliams transform, that is,

$$W_C(x, y) = W_C\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right).$$

It is furthermore called **even** if

$$W_C(x, -y) = W_C(x, y),$$

and **odd** otherwise. The generator matrix of a  $[2n, n]$  code is said to be in **systematic form** if it is blocked as  $(I, A)$  with  $I$  the identity matrix of order  $n$ . If  $A$  is circulant then  $C$  is said to be **double circulant**.

### B. Boolean functions

A permutation  $F$  of  $\mathbb{F}_2^n$  is any bijective map from  $\mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ . Its Walsh-Hadamard transform at  $(a, b)$  is defined as

$$\widehat{F}(a, b) = \sum_{x \in \mathbb{F}_2^n} (-1)^{a \cdot x + b \cdot F(x)},$$

where  $a \cdot x$  denotes the scalar product of vectors  $a$  and  $x$ .

### C. Dual distance

If  $C$  is a binary code, let  $B_i$  denote its distance distribution, that is,

$$B_i = \frac{1}{|C|} |\{(x, y) \in C \times C \mid d(x, y) = i\}|$$

The dual distance distribution  $B_i^\perp$  is the MacWilliams transform of the distance distribution, in the sense that

$$D_C^\perp(x, y) = \frac{1}{|C|} D_C(x + y, x - y),$$

where

$$D_C(x, y) = \sum_{i=0}^n B_i x^{n-i} y^i,$$

and

$$D_C^\perp(x, y) = \sum_{i=0}^n B_i^\perp x^{n-i} y^i.$$

The **dual distance** of  $C$  is the smallest  $i > 0$  such that  $B_i^\perp \neq 0$ . When  $C$  is linear, it is the minimum distance of  $C^\perp$ , since  $D_C^\perp(x, y) = D_{C^\perp}(x, y)$ .

For  $u \in \mathbb{F}_2^n$ , define the **character** attached to  $u$  evaluated in  $C$  as

$$\chi_u(C) = \sum_{v \in C} (-1)^{u \cdot v}.$$

### D. $\mathbb{Z}_4$ -codes

Recall that the Gray map  $\phi$  from  $\mathbb{Z}_4$  to  $\mathbb{F}_2^2$  is defined by

$$\phi(0) = 00, \phi(1) = 01, \phi(2) = 11, \phi(3) = 10.$$

This map is extended componentwise from  $\mathbb{Z}_4^n$  to  $\mathbb{F}_2^{2n}$ . A  $\mathbb{Z}_4$ -code of length  $n$  is a  $\mathbb{Z}_4$ -submodule of  $\mathbb{Z}_4^n$ . The **binary image**  $\phi(C)$  of a  $\mathbb{Z}_4$ -code  $C$  is just  $\{\phi(c) \mid c \in C\}$ . In general a  $\mathbb{Z}_4$ -code  $C$  is of type  $4^k 2^l$  if  $C \approx \mathbb{Z}_4^k \mathbb{Z}_2^l$  as additive groups. A  $\mathbb{Z}_4$ -code is called **free** if  $l = 0$ . An important class of  $\mathbb{Z}_4$ -codes is that of  $QR(p+1)$  where QR stands for **Quadratic Residue codes** and  $p$  is a prime congruent to  $\pm 1$  modulo 8. They were introduced as extended cyclic codes in [2], based on Hensel lifting of classical binary quadratic residue codes [20].

## III. GENERALIZATION TO NON-LINEAR CODES

Recall that in the context of [22], [23] we need to use Boolean permutations - more precisely, permutations  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ , such that, given some integer  $d$  as large as possible, for every pair of vectors  $a, b \in \mathbb{F}_2^n$  such that the concatenated  $(a, b)$  is nonzero and has Hamming weight  $< d$ , the value of the Walsh-Hadamard transform of  $F$  at  $(a, b)$ , is null. Such Boolean functions are called **d-GCI**, for **Graph Correlation Immune**.

In [23] the following proposition is proven:

**Proposition III.1.** *The existence of a linear d-GCI function of  $n$  variables is equivalent to the existence of a CIS code of parameters  $[2n, n, \geq d]$  with the systematic partition.*

To generalize this result to systematic, possibly nonlinear codes, we attach to such a vectorial function  $F$  the code  $C_F$  defined by the relationship

$$C_F = \{(x, F(x)) \mid x \in \mathbb{F}_2^n\}.$$

**Theorem III.2.** *The permutation  $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$  is a  $d$ -GCI function of  $n$  variables if and only if the code  $C_F$  has dual distance  $\geq d$ .*

**Proof.** The proof follows immediately by the characterization of the dual distance of a code  $C$  in terms of characters  $\chi_u(C)$  of  $C$  regarded as an element in the group algebra  $\mathbb{Q}[\mathbb{F}_2]$  [20, Chap. 5, Theorem 7]. Essentially, this characterization says that  $d^\perp$  is the smallest non zero weight of a vector  $u \in \mathbb{F}_2^n$  such that  $\chi_u(C) \neq 0$ . Note that the value of the Walsh-Hadamard transform of  $F$  at  $(a, b)$  is  $\chi_u(C)$  for  $u = (a, b)$  and  $C = C_F$ . ■

Since the dual of a CIS code is also CIS, Proposition III.1 follows at once from Theorem III.2. This theorem also allows us to construct better GCI functions by using nonlinear codes. Define a free  $\mathbb{Z}_4$ -code of length  $n$  with  $2^n$  codewords to be **CIS** if it contains two disjoint information sets.

**Theorem III.3.** *If  $\mathcal{C}$  is a free systematic  $\mathbb{Z}_4$ -code of length  $n$  with  $2^n$  codewords, then its binary image is a systematic code of the form  $C_F$  for some  $F$ . Furthermore,  $\mathcal{C}$  is CIS with systematic partition if and only if  $F$  is one-to-one.*

**Proof.** By hypothesis the generator matrix of  $\mathcal{C}$  is of the shape  $(I, A)$  and therefore any codeword of  $\mathcal{C}$  can be cast as  $(x, \mathcal{F}(x))$  for some  $\mathbb{Z}_4$  linear map  $\mathcal{F}$  from  $\mathbb{Z}_4^n$  to  $\mathbb{Z}_4^n$ . The result follows by taking the binary image of each codeword. The CIS property entails that  $\mathcal{F}$  is bijective, and, consequently, so is  $F$ . Conversely, if  $F$  is bijective then  $[n + 1..2n]$  is an information set. ■

**Example III.4.** Consider the Nordstrom Robinson code in length 16. It is the image of the octacode [12], which is free and CIS since it is self-dual. It therefore satisfies the hypothesis of Theorem III.3 and therefore can be attached to a 6-GCI function in 8 variables, when the best linear CIS code only gives a 5-GCI function as shown in the next section.

**Example III.5.** Consider  $QR32$  a self-dual extended cyclic  $\mathbb{Z}_4$ -code [3]. Its binary image of length 64 has distance 14, which is better than the best known  $[64, 32]$  binary code of distance 12. Similarly,  $QR48$  has a binary image of distance 18 [3], when the best binary rate one-half code of length 96 has distance 16.

**Example III.6.** Recently, Kiermaier and Wassermann [15] have computed the Lee weight enumerator of the type II  $\mathbb{Z}_4$ -code  $QR80$  and its minimum Lee weight  $d_L = 26$ . Hence its binary image has distance 26, which is better than the best known  $[160, 80]$  binary code of distance 24.

#### IV. GENERAL CONSTRUCTIONS FOR CIS CODES

We now consider general constructions for the class of CIS codes. The following two lemmas are, respectively, immediate and well-known.

**Lemma IV.1.** *If a  $[2n, n]$  code  $C$  has generator matrix  $(I, A)$  with  $A$  invertible then  $C$  is CIS with the systematic partition. Conversely, every CIS code is equivalent to a code with a generator matrix in that form.*

In particular this lemma applies to systematic self-dual codes whose generator matrix  $(I, A)$  satisfies  $AA^T = I$ . In the following two results, we identify circulant matrices with the polynomial in  $x$  whose  $x$ -expansion coincides with the first row of the matrix.

**Lemma IV.2.** *Let  $f(x)$  be a polynomial over  $\mathbb{F}_2$  of degree less than  $n$ . Then,  $\gcd(f(x), x^n - 1) = 1$  if and only if the circulant matrix generated by  $f(x)$  has  $\mathbb{F}_2$ -rank  $n$ .*

**Proposition IV.3.** *The double circulant code whose generator matrix is represented by  $(1, f(x))$  satisfying Lemma IV.2 is a CIS code.*

**Proof.** Combine Lemma IV.1 with Lemma IV.2. ■

In the other direction, a partial converse to Lemma IV.1 is the following.

**Proposition IV.4.** *If a  $[2n, n]$  code  $C$  has generator matrix  $(I, A)$  with  $rk(A) < n/2$  then  $C$  is not CIS.*

**Proof.** Let  $L = [1..n]$ , and  $R = [n + 1..2n]$ . If  $I$  is an information set, by the rank hypothesis, then  $|I \cap R| < n/2$  and, consequently,  $|I \cap L| > n/2$ . So two distinct information sets must intersect non trivially in  $L$ . ■

The next proposition is an immediate but useful observation.

**Proposition IV.5.** *If  $C$  is a  $[2n, n]$  code whose dual has minimum weight 1 then  $C$  is not CIS.*

**Proof.** If the dual of  $C$  has minimum weight 1 then the code  $C$  has a zero column and therefore cannot be CIS. ■

The previous proposition permits to show it is possible for an optimal code not to be CIS:

**Proposition IV.6.** *There exists at least one optimal binary code that is not CIS.*

**Proof.** The  $[34, 17, 8]$  code described in the Magma package  $BKLC(GF(2), 34, 17)$  (best known linear code of length 34 and dimension 17) is an optimal code (minimum weight 8 is the best possible minimum distance for such a code) whose dual has minimum distance 1, and therefore is not CIS. ■

A special class of CIS codes is obtained from combinatorial matrices [6]. Let  $A$  be an integral matrix with 0/1 valued entries. We shall say that  $A$  is the adjacency matrix of a **strongly regular graph** (SRG) of parameters  $(n, \kappa, \lambda, \mu)$  if  $A$  is symmetric, of order  $n$ , verifies  $AJ = JA = \kappa J$  and satisfies

$$A^2 = \kappa I + \lambda A + \mu(J - I - A),$$

where  $I, J$  are the identity and all-one matrices of order  $n$ . Alternatively we shall say that  $A$  is the adjacency matrix of a **doubly regular tournament** (DRT) of parameters  $(n, \kappa, \lambda, \mu)$  if  $A$  is skew-symmetric, of order  $n$ , verifies  $AJ = JA = \kappa J$  and satisfies

$$A^2 = \lambda A + \mu(J - I - A).$$

In the next result we identify  $A$  with its reduction mod 2.

**Proposition IV.7.** *Let  $C$  be the linear binary code of length  $2n$  spanned by the rows of  $(I, M)$ . With the above notation,  $C$  is CIS if  $A$  is the adjacency matrix of a*

- *SRG of odd order with  $\kappa, \lambda$  both even and  $\mu$  odd and if  $M = A + I$*
- *DRT of odd order with  $\kappa, \mu$  odd and  $\lambda$  even and if  $M = A$*

- SRG of odd order with  $\kappa$  even and  $\lambda, \mu$  both odd and if  $M = A + J$
- DRT of odd order with  $\kappa$  even and  $\lambda, \mu$  both odd and if  $M = A + J$

**Proof.** In the first case, reduce the quadratic matrix equation modulo 2 to obtain

$$A^2 = J + I + A.$$

If  $x \in \mathbb{F}_2^n$  is non trivial in  $\text{Ker}(A + I)$ , then the above equation written as  $A(A + I) = J + I$ , entails  $Jx = x$  which implies, by studying the eigenspaces of  $J$  that  $x$  is the all one vector  $\mathbf{1}$ . But because  $\kappa$  is even we know that  $A\mathbf{1} = 0$ . This is a contradiction. So,  $\text{Ker}(M)$  is trivial and the result follows by Lemma IV.1. The proof in the second case is analogous. In the third case the matrix equation becomes

$$A^2 = J + I.$$

If  $x \in \text{Ker}(M)$  then  $x \in \text{Ker}(M^2)$ , but, by the hypotheses on  $\kappa$  and  $n$ , we see that  $M^2 = A^2 + J = I$ , yielding  $x = 0$ . The proof of the fourth case is analogous to that of the third case. ■

Let  $q$  be an odd prime power. Let  $Q$  be the  $q$  by  $q$  matrix with zero diagonal and  $q_{ij} = 1$  if  $j - i$  is a square in  $GF(q)$  and zero otherwise.

**Corollary IV.8.** *If  $q = 8j + 5$  then the span of  $(I, Q + I)$  is CIS. If  $q = 8j + 3$  then the span of  $(I, Q)$  is CIS.*

**Proof.** It is well-known [6] that  $q = 4k + 1$  then  $Q$  is the adjacency matrix of a SRG with parameters  $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$ . If  $q = 4k + 3$  then  $Q$  is the adjacency matrix of a DRT with parameters  $(q, \frac{q-1}{2}, \frac{q-3}{4}, \frac{q+1}{4})$ . The result follows by Proposition IV.7. ■

The codes obtained in that way are Quadratic Double Circulant codes [8]. The third and fourth cases of Prop IV.7 cannot apply to either Paley SRG or DRT since for these we have  $\mu - \lambda = 1$ .

**Example IV.9.** Let  $A$  be the adjacency matrix of an SRG of parameters  $(57, 24, 11, 9)$ , which exists by Brouwer's database [4]. By the third case of Proposition IV.7 we get a CIS code of length 114.

Now we look at CIS codes constructed from **cyclic codes**. Denote by  $C_i$  the code  $C$  shortened at coordinate  $i$  and by  $\overline{C}$  the extension of  $C$  by an overall parity check.

**Proposition IV.10.** *Let  $C$  be a cyclic binary code of odd length  $N$ , and dimension  $\frac{N+1}{2}$ . If its generator matrix is in circulant form, both  $C_1$  and  $C_N$  are CIS with the systematic partition. If, furthermore, the weight of the generator polynomial is odd, then  $\overline{C}$  is CIS with the systematic partition.*

**Proof.** Recall that in a cyclic code of dimension  $k$ , consecutive  $k$  indices form an information set. The result follows then for  $C_1$  and  $C_N$ . In the extended case, the generator matrix of  $\overline{C}$  is obtained from that of  $C$  by inserting a column of 1's in position  $\frac{N+3}{2}$ . It consists then of two juxtaposed upper triangular and lower triangular, non singular matrices. ■

## V. CIS CODES WITH RECORD DISTANCES

In this section, we construct optimal or best-known CIS codes of length  $2n \leq 130$ . We refer to [14] for the highest known minimum distances of rate one-half codes for lengths up

to 48. In the rest of the section we list what we know of CIS codes of length  $< 130$ . All statements referring to best rate one-half codes of lengths  $> 48$  come from [28]. All statements on existence of self-dual codes are from either [9] or [10]. We could have used [11]. The command  $BKLC(GF(2), n, k)$  from the computer package Magma [26] means the best known binary linear  $[n, k]$  code as per [28]. We put a \* as an exponent of a distance if the CIS code is optimal as a rate one-half code. The table captions are as follows

- bk= obtained the command  $BKLC(GF(2), n, k)$  from Magma.
- dc=double circulant
- fsd=formally self-dual
- id=isodual
- nsd= not self-dual
- nfsd= not formally self-dual
- qdc=quadratic double circulant
- sc=special construction
- sd= self-dual
- sqr=shortened quadratic residue code
- xqdc=extended quadratic double circulant
- xqr=extended quadratic residue code

#### A. Lengths 2 to 32

$2n$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
$d$	2*	2*	3*	4*	4*	4*	4*	5*	6*	6*	7*	8*	7*	8*	8*	8*
code	dc	dc	$\sim$ dc	sd	dc	sd	sd	$\sim$ dc	$\sim$ dc	nfsd	id	sd	fsd	dc	dc	sd

- $2n = 2, 4$ . Double circulant codes of length  $2n = 2, 4$  with first rows 1 or 10, respectively, are optimal CIS codes with minimum distance 2. They are in fact self-dual.
- $2n = 6$ . Lemma IV.2 does not imply that double circulant codes of this length with  $d = 3$  are CIS codes. However, the equivalent code  $\{(100\ 011), (010\ 101), (001\ 111)\}$  can be checked to be an optimal CIS code with  $d = 3$ .
- $2n = 8$ . The extended Hamming  $[8, 4, 4]$  code is an optimal CIS code.
- $2n = 10$ . The double circulant with first row 0111 in [14] is an optimal CIS code with  $d = 4$  by Lemma IV.2.
- $2n = 12$ . There exists a self-dual  $[12, 6, 4]$  code, which is an optimal CIS code.
- $2n = 14$ . There exists a self-dual  $[14, 7, 4]$  code, which is an optimal CIS code.
- $2n = 16$ . There is a unique optimal  $[16, 8, 5]$  code [14], which is also odd formally self-dual. This is a double circulant code with first row 00010111. It cannot be a CIS code with systematic partition by Lemma IV.2. We need a new representation of this code. Take  $C_{16}$  as  $BKLC(GF(2), 16, 8)$  in Magma. We check that  $C_{16}$  is also an odd formally self-dual  $[16, 8, 5]$  code. The determinant of the right half submatrix of the standard generator matrix of  $C_{16}$  is 1. Hence  $C_{16}$  is an optimal CIS code.
- $2n = 18$ . There is a unique optimal  $[18, 9, 6]$  code [14], which is also odd formally self-dual. It is described as a double circulant code with first row 001001111. The corresponding polynomial  $x^6 + x^3 + x^2 + x + 1$  is factored as  $(x^2 + x + 1)(x^4 + x^3 + 1)$ . As  $x^9 + 1$  has



a factor  $x^3 + 1 = (x + 1)(x^2 + x + 1)$ , this code cannot be proved to be CIS by Lemma IV.2. On the other hand, we take  $C_{18}$  as  $BKLC(GF(2), 18, 9)$  in Magma, which is also a  $[18, 9, 6]$  code. The determinant of the right half submatrix of the standard generator matrix of  $C_{18}$  is 1. Hence  $C_{18}$  is an optimal CIS code.

- $2n = 20$ . There are 1682 optimal  $[20, 10, 6]$  codes [14]. There are exactly eight formally self-dual codes among them [14]. We obtain the first non-formally self-dual optimal CIS code by taking  $BKLC(GF(2), 20, 10)$  in Magma notation with the systematic partition.
- $2n = 22$ . We take  $BKLC(GF(2), 20, 10)$  in Magma notation with the systematic partition. This code is isodual and fsd.
- $2n = 24$ . The extended Golay code is CIS as a self-dual code. It is also optimal as a rate one-half code.
- $2n = 26$ . The fsd isodual code  $C_{26,1}$  in the notation of [14] is CIS with the systematic partition.
- $2n = 28$ . The even fsd  $BKLC(GF(2), 20, 10)$  is not self-dual but still CIS with the systematic partition by Lemma IV.1.
- $2n = 30$ . We use a double circulant code with generator matrix  $(1, f)$  where  $f = x^{10} + x^8 + x^7 + x^5 + x^3 + x + 1$ , an irreducible polynomial. This code is CIS with the systematic partition by Proposition IV.3.
- $2n = 32$ . There are five extremal Type II  $[32, 16, 8]$  self-dual codes. They are also optimal as rate one-half codes.

### B. Lengths 34 to 64

$2n$	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62	64
$d$	8*	8*	8	9	10*	10*	11*	12*	10	10	12	12	12	12	12	12
code	sc	sd	sd	nsd	fsd	fsd	fsd	sd	sd	sd	bk	sd	qdc	sd	sd	sd

- $2n = 34$ . We have checked that  $BKLC(GF(2), 34, 17)$  of distance 8 is not CIS with systematic partition. There are s-extremal self-dual  $[34, 17, 6]$  codes (see [1]). Taking the doubly-even subcode of such a code and adding an element of the shadow we obtain a  $[34, 17, 8]$  code with generator matrix  $(I, A)$  with  $A$  given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Then it can be checked that this code is CIS with partition  $L = \{14, \dots, 30\}$  and  $R = \{1, \dots, 13, 31, \dots, 34\}$ .

- $2n = 36$ . There are many self-dual Type I  $[36, 18, 8]$  self-dual codes. They are also optimal as rate one-half codes.

- $2n = 38$ . There are many self-dual Type I  $[38, 19, 8]$  self-dual codes. They are the best known rate one-half codes of that length.
- $2n = 40$ . There is an odd fsd isodual  $BKLC(GF(2), 40, 20)$  of distance 9, the best known rate one-half code of that length. It is CIS by computer check of Lemma IV.1.
- $2n = 42$ . There is an even fsd isodual  $BKLC(GF(2), 42, 21)$  of distance 10, an optimal rate one-half code of that length. It is CIS by computer check of Lemma IV.1.
- $2n = 44$ . There is an odd fsd isodual  $BKLC(GF(2), 44, 22)$  of distance 10, an optimal rate one-half code of that length. It is CIS by computer check of Lemma IV.1.
- $2n = 46$ . There is an odd fsd isodual  $BKLC(GF(2), 46, 23)$  of distance 11, an optimal rate one-half code of that length. It is CIS by computer check of Lemma IV.1.
- $2n = 48$ . There is a unique Type II  $[48, 24, 12]$  code, an optimal rate one-half code of that length.
- $2n = 50$ . There are at least 6 Type I self-dual codes of distance 10, which is best known as per [28].
- $2n = 52$ . There are at least 499 Type I self-dual codes of distance 10, which is best known as per [28].
- $2n = 54$ . The code  $BKLC(GF(2), 54, 27)$  has distance 11. Computing a determinant shows that it is CIS with the systematic partition.
- $2n = 56$ . There are Type II self-dual codes of distance 12, which is the best known distance for rate one-half codes of that length.
- $2n = 58$ . The Quadratic Double Circulant attached to the prime 29 has distance 12 and is CIS with the systematic partition by determinant computation.
- $2n = 60$ . There are at least 15 Type I self-dual codes of distance 12, the best known distance for rate one-half codes of that length.
- $2n = 62$ . There are at least 20 Type I self-dual codes of distance 12, the best known distance for rate one-half codes of that length.
- $2n = 64$ . There are at least 3270 Type II self-dual codes of distance 12, the best known distance for rate one-half codes of that length.

### C. Lengths 66 to 100

$2n$	66	68	70	72	74	76	78	80	82	84	86	88	90	92	94	96	98	100
$d$	12	13	15	15	14	14	15	16	14	15	16	17	18	16	16	16	17	18
code	sd	bk	bk	bk	xqr	sd	bk	sd	sd	bk	sd	sqr	bk	sd	sd	sd	bk	sc

- $2n = 66$ . There are at least 3 Type I self-dual codes of distance 12, the best known distance for rate one-half codes of that length.
- $2n = 68$ . The code  $BKLC(GF(2), 68, 34)$  of distance 13 is CIS with the systematic partition, by determinant computation.
- $2n = 70$ . The code  $BKLC(GF(2), 70, 35)$  of distance 15 is CIS with the systematic partition, by determinant computation.
- $2n = 72$ . The code  $BKLC(GF(2), 72, 36)$  of distance 15 is CIS with the systematic partition, by determinant computation.
- $2n = 74$ . Is  $BKLC(GF(2), 74, 37)$  of distance 14 CIS? The extended quadratic residue code  $[74, 37, 14]$  is CIS with the systematic partition by Proposition IV.10.

- $2n = 76$ . There are at least 2 Type I self-dual codes of distance 14, the best known distance for rate one-half codes of that length.
- $2n = 78$ . A shortened code of quadratic residue code  $[79, 40, 15]$  leads to a  $[78, 39, 14]$  CIS code with the systematic partition, by Proposition IV.10.
- $2n = 80$ . There are at least 15 Type II self-dual codes of distance 16, the best known distance for rate one-half codes of that length.
- $2n = 82$ . There is at least 1 Type I self-dual code of distance 14, the best known distance for rate one-half codes of that length.
- $2n = 84$ . The code  $BKLC(GF(2), 84, 42)$  of distance 15 is CIS with the systematic partition, by determinant computation.
- $2n = 86$ . There is at least 1 Type I self-dual code of distance 16, the best known distance for rate one-half codes of that length.
- $2n = 88$ . A shortened code of the quadratic residue code  $[89, 45, 17]$  leads to a  $[88, 44, 17]$  CIS code with the systematic partition, by Proposition IV.10.
- $2n = 90$ . The code  $BKLC(GF(2), 88, 44)$  of distance 18 is CIS with the systematic partition, by determinant computation.
- $2n = 92$ . There are at least 1 Type I self-dual code of distance 16, the best known distance for rate one-half codes of that length.
- $2n = 94$ . There is at least 1 Type I self-dual code of distance 16, the best known distance for rate one-half codes of that length [13].
- $2n = 96$ . There is at least 1 Type I self-dual code of distance 16, the best known distance for rate one-half codes of that length.
- $2n = 98$ . The code  $BKLC(GF(2), 98, 49)$  of distance 17 is CIS with the systematic partition, by determinant computation.
- $2n = 100$ . The code  $BKLC(GF(2), 100, 50)$  of distance 18 is CIS with the systematic partition, since it is obtained by puncturing and double shortening from a Quadratic Residue code of length 103 [28]. The argument is similar to Proposition IV.10.

#### D. Lengths 102 to 130

$2n$	102	104	106	108	110	112	114	116	118	120	122	124	126	128	130
$d$	19	20	19	20	18	19	20	20	20	20	20	21	21	22	22
code	bk	bk	qdc	bk	bk	bk	bk	sc	sc	sd	qdc	xqdc	bk	bk	sc

- $2n = 102$ . The code  $BKLC(GF(2), 102, 51)$  of distance 19 is CIS with the systematic partition, by determinant computation.
- $2n = 104$ . The code  $BKLC(GF(2), 104, 52)$  of distance 20 is CIS with the systematic partition, by determinant computation.
- $2n = 106$ . The code  $BKLC(GF(2), 106, 53)$  of distance 19 is CIS by computer search.
- $2n = 108$ . The code  $BKLC(GF(2), 108, 54)$  of distance 20 is CIS with the systematic partition, by determinant computation.
- $2n = 110$ . The code  $BKLC(GF(2), 110, 55)$  of distance 18 is CIS with the systematic partition, by determinant computation.
- $2n = 112$ . The code  $BKLC(GF(2), 112, 56)$  of distance 19 is CIS with the systematic partition, by determinant computation.

- $2n = 114$ . The code  $BKLC(GF(2), 114, 57)$  of distance 20 is CIS with the systematic partition, by determinant computation.
- $2n = 116$ . The code  $BKLC(GF(2), 116, 58)$  is not CIS by Prop IV.5, since its dual has minimum weight 1. A CIS code can be obtained in the following way: consider  $g(x)$  the generator polynomial of the  $[127, 71, 19]$  BCH code of length 127 with designed distance 19. Consider a generator matrix  $G_0$  of the code obtained by shifting the coefficients of  $g(x)$  in the usual way. Consider now the matrices  $G_1$  and  $G_2$  obtained by shifting  $G_0$  of respectively 1 and 2 positions. We now construct the matrix  $G' = G_0 + G_1 + G_2$ . We then add to  $G'$  a last column of 1's and we erase the first row to obtain a matrix  $G$ . The matrix  $G$  generates a  $[128, 70, 20]$  code  $C$ . Now if one shortens  $C$  on the first twelve columns, one obtains a  $[116, 58, 20]$  code which is CIS with the systematic partition.
- $2n = 118$ . The code  $BKLC(GF(2), 118, 59)$  is not CIS by Prop IV.5, since its dual has minimum weight 1. Similarly to the case  $2n = 116$ , a CIS code can be obtained in the following way: consider  $g(x)$  the generator polynomial of the BCH code of length 127 and designed distance 19. Consider a generator matrix  $G_0$  of the code obtained by shifting the coefficients of  $g(x)$  in the usual way. Consider now the matrices  $G_1$  and  $G_2$  obtained by shifting  $G_0$  of respectively 1 and 2 positions. We now construct the matrix  $G' = G_0 + G_1 + G_2$ . We then add to  $G'$  a last column of 1 and we erase the first two rows to obtain a matrix  $G$ . The matrix  $G$  generates a  $[128, 69, 20]$  code  $C$ . Now if one shortens the first ten columns of  $C$ , one obtains a  $[118, 59, 20]$  code which is CIS with the systematic partition.
- $2n = 120$ . There is at least 1 Type II self-dual code of distance 20, the best known distance for rate one-half codes of that length.
- $2n = 122$ . Is  $BKLC(GF(2), 122, 61)$  of distance 20 CIS? The code  $QDC(61)$  of distance 20 is CIS with the systematic partition, by determinant computation.
- $2n = 124$ . Is  $BKLC(GF(2), 124, 62)$  of distance 20 CIS? The code

*ExtendCode(BorderedDoublyCirculantQRCode(61))*

of distance 20 is CIS with the systematic partition, by determinant computation.

- $2n = 126$ . The code  $BKLC(GF(2), 126, 63)$  of distance 21 is CIS with the systematic partition, by determinant computation.
- $2n = 128$ . The code  $BKLC(GF(2), 128, 64)$  of distance 22 is CIS with the systematic partition, by determinant computation.
- $2n = 130$ . The code  $BKLC(GF(2), 130, 65)$  of distance 22 is CIS by computer search.

## VI. CLASSIFICATION

### A. Number of equivalence classes of CIS codes

Let  $n \geq 2$  be an integer and  $g_n$  denote the cardinality of  $GL(n, 2)$  the general linear group of dimension  $n$  over  $GF(2)$ . It is well-known (see [20, p.399]), that

$$g_n = \prod_{j=0}^{n-1} (2^n - 2^j).$$

**Proposition VI.1.** *The number  $e_n$  of equivalence classes of CIS codes of dimension  $n \geq 2$  is at most  $g_n/n!$ .*

**Proof.** By Lemma IV.1 every CIS code of dimension  $n$  is equivalent to the linear span of  $(I, A)$  for some  $A \in GL(n, 2)$ . But the columns of such an  $A$  are pairwise linearly independent, hence pairwise distinct. Permuting the columns of  $A$  leads to equivalent codes. ■

**Example VI.2.** There is a unique CIS code in length 2 namely  $R_2$  the repetition code of length 2. For  $n = 2$ , the  $g_2 = 6$  invertible matrices reduce to three under column permutation: the identity matrix  $I$  and the two triangular matrices  $T_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ , and  $T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ . The generator matrix  $(I, I)$  spans the direct sum  $R_2 \oplus R_2$ , while the two codes spanned by  $(I, T_1)$  and  $(I, T_2)$  are equivalent to a code  $C_3$ , an isodual code which is not self-dual. Thus  $e_2 = 2 < g_2/2! = 3$ .

The numbers  $g_n/n!$  grow very fast: 3, 28, 840, 83328. They count the number of bases of  $\mathbb{F}_2^n$  over  $\mathbb{F}_2$  [27, A053601]. The numbers  $e_n$  do not grow so fast as can be seen by looking at Table I.

### B. Building up construction

The building up construction [16] is known for binary self-dual codes. In this section, we extend it to CIS codes. We show that every CIS code can be constructed in this way.

**Lemma VI.3.** *Given a  $[2n, n]$  CIS code  $C$  with generator matrix  $(I_n|A)$  where  $A$  is an invertible square matrix of order  $n$ , we can obtain a  $[2(n-1), n-1]$  CIS code  $C'$  with generator matrix  $(I_{n-1}|A')$ , where  $A'$  is an invertible square matrix of order  $n-1$ .*

**Proof.** Let  $a_i$  be the  $i$ th column of  $A$  and  $r_i$  be the  $i$ th row of  $A$  for  $1 \leq i \leq n$ . We delete the first column  $a_1$ . Then for each  $i$ , let  $r'_i$  be the  $i$ th row obtained from  $r_i$  after the removal of the first coordinate of  $r_i$ . Since there are  $n$  rows  $r'_i$  in the  $(n-1)$ -dimensional space, there is  $j$  such that  $r'_j = \sum_{i \neq j} c_i r'_i$ , where  $c_i$  is uniquely determined. Then delete the  $j$ th row of  $(I_n|A)$  and the  $j$ th column of  $(I_n|A)$  to get a  $[2(n-1), n-1]$  CIS code  $C'$  with generator matrix  $(I_{n-1}|A')$ , where  $A'$  is invertible. We remark that  $A'$  is a square matrix of order  $n-1$  whose rows consist of the  $n-1$   $r'_i$ 's except for  $r'_j$  and whose columns consist of  $a'_i$ 's ( $2 \leq i \leq n$ ), each of which is the  $i$ th column obtained from  $a_i$  after the removal of the  $j$ th component of  $a_i$ . ■

**Proposition VI.4** (Building up construction). *Suppose that  $C$  is a  $[2n, n]$  CIS code  $C$  with generator matrix  $(I_n|A)$ , where  $A$  is invertible with  $n$  rows  $r_1, \dots, r_n$ . Then for any two vectors  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)^T$  the following matrix  $G_1$  generates a  $[2(n+1), n+1]$  CIS code  $C_1$  with the systematic partition:*

$$G_1 = \left( \begin{array}{c|ccc|c|c} 1 & 0 & 0 & \cdots & 0 & z_1 & x \\ 0 & 1 & 0 & \cdots & 0 & y_1 & r_1 \\ 0 & 0 & 1 & \cdots & 0 & y_2 & r_2 \\ \vdots & & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & y_n & r_n \end{array} \right), \quad (1)$$

where there are multipliers  $c_i$ 's satisfying  $x = \sum_{i=1}^n c_i r_i$  and  $z_1 = 1 + \sum_{i=1}^n c_i y_i$ .

**Proof.** It suffices to show that the rows of the right half of  $G_1$  are linearly independent. Suppose  $\alpha(z_1|x) + \beta_1(y_1|r_1) + \cdots + \beta_n(y_n|r_n) = 0$ . Then  $\alpha x + \sum_{i=1}^n \beta_i r_i = 0$ . If  $\alpha = 0$ , then  $\beta = 0$  for all  $i$  as it should. If  $\alpha = 1$ , then  $x + \sum_{i=1}^n \beta_i r_i = 0$ . Since  $x = \sum_{i=1}^n c_i r_i$  for unique  $c_i$ 's, we

have  $\beta_i = c_i$  for all  $i$ . Thus  $0 = z_1 + \sum_{i=1}^n c_i y_i = 1 + \sum_{i=1}^n c_i y_i + \sum_{i=1}^n c_i y_i = 1$ , a contradiction. Thus the rows of the right half of  $G'$  are linearly independent ■

**Example VI.5.** Let us consider a  $[6, 3, 3]$  CIS code  $C$  whose generator matrix is given below.

$$G = \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right).$$

In order to apply Proposition VI.4, we take for example  $x = (1, 1, 0)$  and  $y = (1, 1, 0)^T$ . Then  $c_1 = c_2 = 1, c_3 = 0$ . Hence  $z = 1$ . In fact, we get the extended Hamming  $[8, 4, 4]$  code whose generator matrix is given below.

$$G_1 = \left( \begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array} \right).$$

Furthermore, starting from  $G_1$ , we can obtain  $G$  back by following the proof of Lemma VI.3.

**Proposition VI.6.** Any  $[2n, n]$  CIS code  $C$  is equivalent to a  $[2n, n]$  CIS code with the systematic partition which is constructed from a  $[2(n-1), n-1]$  CIS code by using Proposition VI.4.

**Proof.** Up to permutation equivalence, we may assume that  $C$  is a  $[2n, n]$  CIS code with systematic generator matrix  $G_2 = (I_n|A)$ , where  $A$  is invertible. For each  $i$  ( $1 \leq i \leq n$ ), we let  $a_i$  be the  $i$ th column of  $A$  and  $r_i$  be the  $i$ th row of  $A$ . It suffices to show that there exists a  $2(n-1) \times (n-1)$  systematic generator matrix  $(I_{n-1}|A')$  with  $A'$  being invertible, whose extension by Proposition VI.4 produces the matrix  $(I_n|A)$  back. We know from Lemma VI.3 that there exists a  $[2(n-1), n-1]$  CIS code  $C'$  with systematic generator matrix  $(I_{n-1}|A')$ , where  $A'$  is invertible. By the remark at the end of the proof of Lemma VI.3,  $A'$  is a square matrix of order  $n-1$  whose rows consist of the  $n-1$   $r'_i$ 's except for  $r'_j$  and whose columns consist of  $a'_i$ 's ( $2 \leq i \leq n$ ), each of which is the  $i$ th column obtained from  $a_i$  after the removal of the  $j$ th component of  $a_i$ . We denote the first column  $a_1$  of  $A$  by  $a_1 = (a_1^1, a_1^2, \dots, a_1^j, \dots, a_1^n)^T$  and  $a'_1$  by the column from  $a_1$  after the removal of the  $j$ th component of  $a_1$ . We choose  $y = a'_1$  and  $x = r'_j$ . Then it follows from Proposition VI.4 that the below generator matrix generates a  $[2n, n]$  CIS code.

$$G_3 = \left( \begin{array}{cccc|ccc|cc} 1 & 0 & 0 & \cdots & 0 & z_1 & r'_j \\ 0 & 1 & 0 & \cdots & 0 & a_1^1 & r'_1 \\ 0 & 0 & 1 & \cdots & 0 & a_1^2 & r'_2 \\ \vdots & & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_1^n & r'_n \end{array} \right),$$

where  $z_1 = 1 + \sum_{i \neq j} c_i a_1^i$  and  $r'_j = \sum_{i \neq j} c_i r'_i$  for some  $c_i$ 's. We claim that this  $z_1$  is equal to the missing component  $a_1^j$  of  $a_1$ . Suppose not. Then  $a_1^j = \sum_{i \neq j} c_i a_1^i$ , that is,  $a_1^j + \sum_{i \neq j} c_i a_1^i = 0$ . Then  $(a_1^j | r'_j) + \sum_{i \neq j} c_i (a_1^i | r'_i) = (0 | 0)$ . This is a contradiction since the set  $\{r_1 = (a_1^1 | r'_1), \dots, r_j = (a_1^j | r'_j), \dots, r_n = (a_1^n | r'_n)\}$  is linearly independent. Thus  $z_1 = a_1^j$ . Therefore, the matrix  $G_3$  is

equivalent to the original matrix  $G_2$  after permuting rows and columns of  $G_3$ . This completes the proof. ■

**Corollary VI.7.** *Let  $C$  be a double circulant CIS  $[2n, n]$  code whose generator matrix is of the form  $G = (I|A)$ , where  $A$  is an invertible circulant matrix whose first row  $r_1$  has odd weight. Then the matrix  $G_1$  below generates a CIS code with systematic partition.*

$$G_1 = \left( \begin{array}{c|c|c|c} 1 & 0 \dots 0 & \epsilon & 1 \dots 1 \\ 0 & & 1 & \\ \vdots & I & \vdots & A \\ 0 & & 1 & \end{array} \right),$$

where  $\epsilon = 0$  if  $n$  is odd and  $\epsilon = 1$  otherwise.

**Proof.** This is the extension of  $(I|A)$  using Proposition VI.4 where  $x$  and  $y$  are all one vectors,  $x$  is the sum of all rows of  $A$ , and  $z = \epsilon$ . ■

**Remark VI.8.** Proposition VI.6 implies that one can construct all  $[2n+2, n+1]$  CIS codes with systematic partition from the set of all  $[2n, n]$  CIS codes with systematic partition, many of which may be equivalent via column permutations. Here is a natural question. Given a complete list of inequivalent  $[2n, n]$  CIS codes with systematic partition, do we always construct all  $[2n+2, n+1]$  CIS codes with systematic partition using Proposition VI.4? This may be so in most cases but it may not be true for some cases. Below is a reason.

Let  $C_i$  ( $i = 1, 2$ ) be a  $[2n, n]$  CIS code with generator matrix  $(I|A_i)$ , where  $A_i$  is invertible. Suppose  $C_1$  is equivalent to  $C_2$  under some  $2n$ -column permutation (possibly interchanging some columns of the left half coordinates of  $C_1$  with some columns in the right half). Let  $\mathbf{D}_i$  ( $i = 1, 2$ ) be the set of all CIS codes built from  $(I|A_i)$  by Proposition VI.4. Then it may not be true that  $\mathbf{D}_1$  is equivalent to  $\mathbf{D}_2$ , that is, for any  $C_3 \in \mathbf{D}_1$ , there exists  $C_4 \in \mathbf{D}_2$  such that  $C_3$  is equivalent to  $C_4$  under some  $2n+2$ -column permutation, and vice versa. This is because the equivalence of  $C_1$  and  $C_2$  is via a permutation on  $2n$  columns but Proposition VI.4 is concerned about the right half of the  $(n+1) \times (2n+2)$  matrix in the Equation (1). Therefore, given a complete list of inequivalent CIS codes of length  $2n$  all of which have systematic partitions, the set of the CIS codes of length  $2n+2$  constructed from Proposition VI.4 does not necessarily give a complete list of inequivalent CIS codes of length  $2n+2$  all of which have systematic partitions. In fact, F. Freibert has informed us that a certain set of the 27 inequivalent  $[8, 4]$  CIS codes with systematic partition produce only 194  $[10, 5]$  CIS codes with systematic partition by Proposition VI.4, although there are exactly 195 inequivalent CIS codes of length 10 (See Proposition VI.12).

In what follows, we give a counting formula similar to a mass formula. This is useful in determining whether a list of inequivalent CIS codes is complete. Recall from Sec. VI-A that  $g_n$  denotes the cardinality of  $GL(n, 2)$ .

**Proposition VI.9.** *Let  $n \geq 2$ . Let  $\mathbf{C}$  be the set of all  $[2n, n]$  CIS codes and let  $S_{2n}$  act on  $\mathbf{C}$  as column permutations of the codes in  $\mathbf{C}$ . Let  $C_1, \dots, C_s$  be representatives from every equivalence class of  $\mathbf{C}$  under the action of  $S_{2n}$ . Let  $\mathbf{C}_{sys}$  be the set of all  $[2n, n]$  CIS codes with generator matrix  $(I_n|A)$  with  $A$  invertible. Suppose that each  $C_i \in \mathbf{C}_{sys}$  ( $1 \leq i \leq s$ ). Then we have*

$$g_n = \sum_{j=1}^s |\text{Orb}_{S_{2n}}(C_j) \cap \mathbf{C}_{sys}|, \quad (2)$$

where  $\text{Orb}_{S_{2n}}(C_j)$  denotes the orbit of  $C_j$  under  $S_{2n}$ .

**Proof.** Let  $A \in GL(n, 2)$ . Then each  $A$  gives a unique  $[2n, n]$  CIS code with generator matrix  $(I|A)$ . Therefore  $g_n = |\mathbf{C}_{sys}|$ . We also note that  $\text{Orb}_{S_{2n}}(C_i) \cap \mathbf{C}_{sys}$  and  $\text{Orb}_{S_{2n}}(C_j) \cap \mathbf{C}_{sys}$  are disjoint whenever  $i \neq j$ . Therefore it is enough to show that each CIS code  $C_A$  with generator matrix  $(I|A)$  belongs to  $\text{Orb}_{S_{2n}}(C_j) \cap \mathbf{C}_{sys}$  for a unique  $j$ . Since  $\{C_1, \dots, C_s\}$  is a set of representatives of all CIS codes,  $C_A$  belongs to a unique orbit  $\text{Orb}_{S_{2n}}(C_j)$  for some  $j$ . Clearly  $C_A$  is in  $\mathbf{C}_{sys}$  by definition. Therefore  $C_A$  belongs to a unique  $\text{Orb}_{S_{2n}}(C_j) \cap \mathbf{C}_{sys}$  as desired. ■

### C. Classification of short CIS codes

We classify all CIS codes of lengths up to 12 up to equivalence. We successively apply Proposition VI.4 from the repetition code of length 2 to obtain lists of inequivalent CIS codes of lengths 4, 6, 8, 10, and 12. For lengths up to 10, we have checked directly that these lists satisfy Equation (2). For length 12, checking directly that our list satisfies Equation (2) takes too long. So by inventing equivalence classes among matrices of  $GF(n, 2)$ , F. Freibert [7] has confirmed that our list of inequivalent CIS codes of length 12 is complete.

It is easy to see that any CIS code has minimum distance  $\geq 2$ .

- $2n = 2$ . It is clear that there is a unique CIS code of length 2, the repetition code.
- $2n = 4$ . Applying Proposition VI.4 to the repetition code of length 2, we show that there are exactly two CIS codes of length 4. Their generator matrices are  $(I|A_{2,1})$  and  $(I|A_{2,2})$ , where

$$A_{2,1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_{2,2} = T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

- $2n = 6$ . Using  $A_{2,1}$ , we have exactly six CIS codes of length 6, one of which is an optimal code of minimum distance 3. Similarly using  $A_{2,2}$ , we have exactly five CIS codes of length 6. But these latter codes are equivalent to some of the former codes. Generator matrices of the form  $(I|A_{3,i})$  ( $1 \leq i \leq 6$ ) are given below. Only  $A_{3,i}$  are shown in order.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Hence we have the following.

**Proposition VI.10.** *There are exactly six CIS codes of length 6. Only one code has  $d = 3$  and the rest have  $d = 2$ .*

- $2n = 8$ . From each of  $(I|A_{3,i})$  ( $1 \leq i \leq 6$ ) we have many CIS codes of length 8 and thus consider the equivalence among them. We have the following.

**Proposition VI.11.** *There are exactly 27 CIS codes of length 8. Only one code has  $d = 4$ , three have  $d = 3$ , and the rest have  $d = 2$ .*

- $2n = 10$ . In a similar manner, we show the following.



**Proposition VI.12.** *There are exactly 195 CIS codes of length 10. Four codes have  $d = 4$ , thirty five codes have  $d = 3$ , and the rest have  $d = 2$ .*

- $2n = 12$ . Furthermore, using the 195 CIS codes of length 10, we have classified CIS codes of length 12.

**Proposition VI.13.** *There are exactly 2705 CIS codes of length 12. More precisely, exactly 41 codes have  $d = 4$ , 565 codes have  $d = 3$ , and the rest have  $d = 2$ .*

We summarize our classification in Table I. The  $i$ th column of the table ( $i = 2, 3, 4$ ) represents the number of CIS codes with  $d = i$  and the parenthesis gives the number of CIS codes in the order of sd, non-sd fsd, and non-fsd. The last column denotes the total number of CIS codes of the corresponding length. The actual generator matrices of lengths 8, 10, 12 will be posted in [17].

TABLE I  
CLASSIFICATION OF ALL CIS CODES OF LENGTHS UP TO 12 IN THE ORDER OF SD, NON-SD FSD, AND NON-FSD

$2n$	$d = 2$	$d = 3$	$d = 4$	Total #
2	1 (1+0+0)			1
4	2 (1+1+0)			2
6	5 (1+2+2)	1 (0+1+0)		6
8	22 (1+9+12)	4 (0+2+2)	1 (1+0+0)	27
10	156 (2+40+114)	35 (0+9+26)	4 (0+2+2)	195
12	2099 (2+318+1779)	565 (0+87+478)	41 (1+7+33)	2705

## VII. LONG CIS CODES

In this section we show that there are long CIS codes satisfying the VG bound for rate one-half codes, that is with relative distance  $\geq H^{-1}(1/2) \approx 0.11$ . We do not use the fact that self-dual codes are CIS. There are polynomial-time constructible binary self-dual codes with relative distance  $\approx 0.02$  [24, p.34, Remark 1]. We begin by a well-known fact [20, p.399].

**Lemma VII.1.** *The number of invertible  $n$  by  $n$  matrices is  $\sim c2^{n^2}$ , with  $c \approx 0.29$ .*

Denote by  $B(n, d)$  the number of invertible matrices  $A$  such that  $d$  columns or less of  $(I, A)$  are linearly dependent. A crude upper bound on this function can be derived as follows.

**Lemma VII.2.** *The quantity  $B(n, d)$  is  $\leq M(n, d)$  where*

$$M(n, d) = \sum_{j=2}^d \sum_{t=1}^{j-1} \binom{n}{j-t} \binom{n}{t} t 2^{n(n-1)}.$$

**Proof.** Let  $j$  be the size of the linearly dependent family of column vectors of  $(I, A)$ , with  $j - t$  columns of  $I$  and  $t$  columns of  $A$ . Choose one column amongst  $t$  to be obtained as linear combination of  $j - 1$  others. Neglecting the invertibility of  $A$  we have  $n - 1$  columns of  $A$  to choose freely. ■

Denote by  $H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$  the binary entropy function [20, p.308].

**Lemma VII.3.** *The quantity  $M(n, d)$  is dominated by  $2^{n^2 - n} 2^{2nH(\delta)}$  when  $d \sim 2\delta n$  with  $0 < \delta < 1$ .*

**Proof.** We evaluate the inner sum in  $M(n, d)$  by the Chu-Vandermonde identity

$$\binom{2n}{j} = \sum_{t=0}^j \binom{n}{t} \binom{n}{j-t}.$$

Then, the outer sum

$$\sum_{j=0}^d \binom{2n}{j}$$

is evaluated by standard entropic estimates for binomials [20, p.310]. Note that  $t \leq n$ , a sub-exponential quantity. ■

**Proposition VII.4.** *For each  $\delta$  such that  $H(\delta) < 0.5$  there are long CIS codes of relative distance  $\delta$ .*

**Proof.** Consider  $(I, A)$  as the parity check matrix of the CIS code and combine Lemmas VII.1, VII.3, VII.2 to ensure that, asymptotically,  $|GL(n, 2)| \gg B(n, d)$  showing the existence of a CIS code of distance  $> d$ , for  $n$  large enough. ■

## VIII. CONCLUSION AND OPEN PROBLEMS

In this article, we have introduced a new class of rate one-half binary codes: CIS codes. In length  $2n$  these codes are, when in systematic form, in one-to-one correspondence with linear bijective permutations in  $n$  variables. More generally, bijective permutations correspond to a certain class of systematic codes. The graph correlation immunity of such permutations is exactly the dual distance of the attached code. Free  $\mathbb{Z}_4$  codes of rate one-half can produce such codes by taking binary images. It would be a worthwhile task to create a database of  $\mathbb{Z}_4$  codes in website form on the model of [28]. There should be some good rate one-half free  $\mathbb{Z}_4$ -codes in the lengths 40 to 80.

We have studied the minimum distances of linear CIS codes up to length 130 and we have found that it is possible to construct CIS codes as good as the best known minimum distance of rate one-half codes, and equal to the best possible distance of these codes up to length 36. Using Table I, we see that the first length when there is an optimal CIS code that is not self-dual is 4, and that the first length when an optimal CIS code cannot be self-dual is 6. The first length when an optimal CIS code is not a formally self-dual code is 20. Thus this new class of codes is richer than both self-dual, and formally self-dual codes. While invariant theory is not available to study these codes, a mass formula analogue is Prop. VI.9. We proved that up to length 130, there exist CIS codes with the best known parameters and we also proved that some optimal codes may not be CIS (for instance in length 34), it hence asks the question whether it is possible to find parameters for which CIS codes have a minimum distance strictly lower than the best linear codes? More generally, does the CIS property entails an upper bound on the minimum distance?

Finally, it is worth extending the definition of CIS codes to other fields than  $\mathbb{F}_2$ , and also to rings. One motivation might be the growing field of Generalized Boolean functions, that is with ranges other than  $\mathbb{F}_2$ . Another motivation like in §3 might be to obtain binary CIS codes by some alphabet changing construction.

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